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Hyper-trigonometry of the particle triangle: II

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Abstract
Using a new kinematical description of a quantum three-body problem in hyperspherical coordinates, we generalize the results of I (Matveenko A V and Czerwonko J 2001 *J. Phys. A: Math. Gen.* **34** 9057) and derive two new infinite series of matrix identities interconnecting geometrical angles, particle masses and internal hyperspherical angles.

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1. Introduction

Different fragmentation channels of the three-body system can be naturally treated in hyperspherical coordinates (Avery 1989). Of course, in this case we need three sets of hyperspherical angles and hyperspherical harmonics (HH). Simple interconnecting formulae can be worked out for that purpose, some of them are derived in this paper. We first recall that after separation of the centre-of-mass variables six coordinates remain. Three of them can be taken to be the Euler angles; the corresponding rigid motion of the particle triangle is then represented by Wigner D-functions. The remaining three coordinates are the hyperradius R (length coordinate) and two angles that we will denote by ξ, η; they describe the internal motion in the body-fixed reference system.

For rotational states having the exact value of the total angular momentum J we may have either normal parity states with the corresponding quantum number of the total parity $p = (-1)^J$ (we shall call them $\epsilon = 0$ states) or abnormal parity ones defined by the relation $p = -(-1)^J$ (we shall use $\epsilon = 1$ in this case). Below we shall be able to introduce ϵ in a formal way. The (Jp)-projected Hamiltonian for the system can then be written generally as

$$\mathbf{H}^{Jp} = -\frac{1}{2M}\frac{1}{R^5}\frac{\partial}{\partial R}R^5\frac{\partial}{\partial R} + \frac{[\Lambda^2(\xi,\eta)]^{Jp}_{\hat{\omega}}}{2MR^2} + V. \tag{1}$$

In this expression M is the reduced mass to be defined below; $\left[\Lambda_i^2\right]^{Jp}_{\hat{\omega}}$ is the matrix operator of the grand angular momentum operator utilizing the body-fixed quantization axis $\hat{\omega}$, it depends on two internal angles but accounts for all the kinetic energy in five hyperangles. Recently, we have introduced special minimal Jp subsets of HH as a pure rotational part of primitives in the variational treatment of three-body states of that symmetry. These HH, being eigenfunctions of the matrix operator $\left[\Lambda_i^2\right]_{\hat{\omega}}^{Jp}$, are, of course, vector-column functions in two variables. It happens that we have exactly $J+\epsilon$ linear independent HH of the $J+\epsilon$ length in our subsets for $(J+\epsilon)*(J+\epsilon)$ operator $\left[\Lambda_i^2\right]_{\hat{\omega}}^{Jp}$. This fact allows for easy manipulations with the corresponding HH solution matrices which can be further given in a simple triangular form for a special choice of the body-fixed quantization axis. We have already exploited this approach and derived the product form for the Wigner rotation matrices (Matveenko 1999) and identities including the associated Legendre polynomials of the same angle (Matveenko and Fukuda 1998). We are also pleased to note that Esry *et al* (2001) have independently introduced the same minimal subsets of HH to derive the threshold laws for three-body recombination.

More recently (Matveenko and Czerwonko 2001, I), we have presented matrix identities interconnecting triangle angles, particle masses and internal hyperspherical angles. The variational calculations of some three-body Coulomb systems using these identities have been just published (Matveenko *et al* 2001). The results of I will be generalized in the present paper utilizing the different choice of the body-fixed quantization axis. Otherwise, the structure of the paper is similar to that of I: we give the simplest nontrivial example of our new identities in the introduction; mathematical details are given in section 2; the third section presents new results and section 4 contains our conclusions.

For a system of three particles with masses m_i $(i=1,2,3)$ we have three sets of Jacobi vectors $\{\mathbf{x}_i,\mathbf{y}_i\}$. As the basis in the particle plane we choose the set $\{i=3\}$: the first Jacobi coordinate $\mathbf{x}_3=\mathbf{x}$ to be the vector from particle 2 to particle 1, with the reduced mass $M_3=M$; and the second Jacobi coordinate $\mathbf{y}_3=\mathbf{y}$ from the centre-of-mass of $(1+2)$ to particle 3, with the reduced mass $\mu_3=\mu$. For the reduced masses in the $\{i\}$ channel we have the well-known expressions

$$\frac{1}{M_i}=\frac{1}{m_j}+\frac{1}{m_k}\qquad\frac{1}{\mu_i}=\frac{1}{m_i}+\frac{1}{m_j+m_k}. \tag{2}$$

Accordingly, three mass parameters: μ, M and $\kappa=(m_2-m_1)/(m_1+m_2)$, will be basic in our approach; using them $\{i=2\}$ Jacobi pair (mass-weighted) can be found from the equality

$$\begin{pmatrix}\hat{\mathbf{x}}_2\\ \hat{\mathbf{y}}_2\end{pmatrix}=\begin{pmatrix}-\cos\phi_{23} & -\sin\phi_{23}\\ \sin\phi_{23} & -\cos\phi_{23}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{x}}_3\\ \hat{\mathbf{y}}_3\end{pmatrix} \tag{3}$$

with a notation chain

$$c_2^4=4c/\rho_2^2\qquad c_3^4=1/4c\qquad\sin^2\phi_{23}=1/\rho_2$$

and $c=\mu/4M$, $\rho_2=1+c(1+\kappa)^2$. The transformation (3) includes the orthogonal matrix of the so-called kinematic rotation by the angle ϕ_{23} (Raynal and Revai 1970). The regular hyperspherical angles are defined for the $\{i\}$ channel by

$$\cos\theta_i=(\hat{\mathbf{x}}_i\,\hat{\mathbf{y}}_i)\qquad\tan\alpha_i=\frac{M_i^{1/2}x_i}{\mu_i^{1/2}y_i} \tag{4}$$

and the hyperradius R will be defined by

$$R=\left(x^2+\frac{\mu}{M}y^2\right)^{1/2}. \tag{5}$$

The identities discussed in I were of the type

$$\begin{pmatrix} \cos\theta'_{23} & \sin\theta'_{23} \\ -\sin\theta'_{23} & \cos\theta'_{23} \end{pmatrix} \begin{pmatrix} \sin\alpha_2 & \cos\alpha_2\cos\theta_2 \\ 0 & -\cos\alpha_2\sin\theta_2 \end{pmatrix}$$
$$= \begin{pmatrix} \sin\alpha_3 & \cos\alpha_3\cos\theta_3 \\ 0 & -\cos\alpha_3\sin\theta_3 \end{pmatrix} \begin{pmatrix} -\cos\phi_{23} & \sin\phi_{23} \\ -\sin\phi_{23} & -\cos\phi_{23} \end{pmatrix}. \tag{6}$$

The geometrical background of (6) is: the triangular solution matrix of vector-column HH in the channel $\{i = 2\}$, see the next section for the details, if multiplied from the left by the rotation matrix (changing the quantization axis), equals the equivalent solution matrix but in the channel $\{i = 3\}$, multiplied from the right by the matrix of kinematic rotation. Here, $\theta'_{23} = \arccos(\hat{\mathbf{x}}_2\hat{\mathbf{x}}_3)$ so that we have $\hat{\mathbf{x}}_2$ as the quantization axis before the rotation and $\hat{\mathbf{x}}_3$ after the rotation. Actually, the above relation was shown to be the $(J = 1, p = -1)$ example of the general identity

$$\hat{\mathbf{d}}^{Jp}(\theta'_{23})\|\mathbf{p}(\alpha_2, \theta_2)\|^{Jp} = \|\mathbf{p}(\alpha_3, \theta_3)\|^{Jp}\mathcal{R}^{Jp}(\phi_{23}) \tag{7}$$

where the matrix elements of the parity-projected Wigner $\hat{\mathbf{d}}^{Jp}(\theta)$ matrices are defined by Matveenko and Fukuda (1996) as

$$d^{Jp}_{mm'} = \frac{1}{(1+\delta_{m0})(1+\delta_{m'0})}\left(d^J_{mm'} + p(-1)^{J+m'}d^J_{m,-m'}\right)$$

and where $d^J_{mm'}$ are matrix elements of the usual Wigner-rotation matrix (Varshalovich *et al* 1998). Orthogonal matrices $\mathcal{R}^{Jp}$ were introduced by Raynal and Revai (1970) and $\|\mathbf{p}(\alpha_i, \theta_i)\|^{Jp} = \|\mathbf{y}(\alpha_i, \theta_i)\|^{Jp}_{\hat{\omega}=\mathbf{x}_i}$ are the upper triangle solution matrices with vector-column HH utilizing $\mathbf{x}_i$ as a body-fixed quantization axis (HH solution matrices $\|y_l(\alpha_i, \theta_i)\|^{Jp}_{\hat{\omega}}$ will be defined later for the arbitrary choice of the quantization axis $\hat{\omega}$ in the triangle plane). While the regular Raynal–Revai transformation allows one to interconnect five-dimensional HH expressed in different Jacobi-channel coordinates, we relate (7) by the rotation in the particle plane of two special degenerate subsets of HH. It is worth noting that the derivation of (7) is strongly related to a formal quantum description of a free three-body problem in hyperspherical coordinates (Matveenko 2001).

To compare with (6), the $(J = 1, p = -1)$ example of the matrix identities derived in this paper reads

$$\begin{pmatrix} \cos\theta''_{23} & \sin\theta''_{23} \\ -\sin\theta''_{23} & \cos\theta''_{23} \end{pmatrix} \begin{pmatrix} \sin\alpha_2\cos\theta_2 & \cos\alpha_2 \\ \sin\alpha_2\sin\theta_2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \sin\alpha_3\cos\theta_3 & \cos\alpha_3 \\ \sin\alpha_3\sin\theta_3 & 0 \end{pmatrix} \begin{pmatrix} -\cos\phi_{23} & \sin\phi_{23} \\ -\sin\phi_{23} & -\cos\phi_{23} \end{pmatrix} \tag{8}$$

where now $\cos\theta''_{23} = (\hat{\mathbf{y}}_2\hat{\mathbf{y}}_3)$ so that we have the second Jacobi vector $\hat{\mathbf{y}}_2$ as the quantization axis before the rotation and, respectively, $\hat{\mathbf{y}}_3$ after the rotation. Generalizing (8) for HH of the arbitrary symmetry we will have the matrix identity

$$\hat{\mathbf{d}}^{Jp}(\theta''_{23})\|\bar{\mathbf{p}}(\alpha_2, \theta_2)\|^{Jp} = \|\bar{\mathbf{p}}(\alpha_3, \theta_3)\|^{Jp}\mathcal{R}^{Jp}(\phi_{23}) \tag{9}$$

where $\|\bar{\mathbf{p}}(\alpha_i, \theta_i)\|^{Jp} = \|\mathbf{y}(\alpha_i, \theta_i)\|^{Jp}_{\mathbf{y}_i}$. One may expect that $\|\bar{\mathbf{p}}(\alpha_i, \theta_i)\|^{Jp}$ and $\|\mathbf{p}(\alpha_i, \theta_i)\|^{Jp}$ HH solution matrices are simply connected by the parity preserving rotation in the particle plane. Really, we have a rather peculiar identity that transforms the triangular matrix with respect to the main diagonal into the triangular one with respect to the second one (in Matveenko (1999) we have found that $\hat{\mathbf{d}}^{Jp}(\theta_i)$ does not depend on α, see also below)

$$\hat{\mathbf{d}}^{Jp}(-\theta_i)\|\mathbf{p}(\alpha_i, \theta_i)\|^{Jp} = \|\bar{\mathbf{p}}(\alpha_i, \theta_i)\|^{Jp} \tag{10}$$

which for the $(J = 1,\ p = -1)$ case will be

$$\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}\begin{pmatrix} \sin\alpha_i & \cos\alpha_i\cos\theta_i \\ 0 & -\cos\alpha_i\sin\theta_i \end{pmatrix} = \begin{pmatrix} \sin\alpha_i\cos\theta_i & \cos\alpha_i \\ \sin\alpha_i\sin\theta_3 & 0 \end{pmatrix}. \tag{11}$$

The structure of the $\|\bar{\mathbf{p}}(\alpha_i, \theta_i)\|^{Jp}$ matrix (upper left triangle one) is completely defined by $\|\mathbf{p}(\alpha_i, \theta_i)\|^{Jp}$ (Matveenko 1999) and will be discussed in the next section.

The derivation of identities will be based on the properties of the well-known three-body HH which, if written in the body-fixed frame, can be factorized into an extrinsic part depending on three Euler-rotation angles and an intrinsic one that depends on two internal variables

$$\mathbf{Y}_{KlL}^{JpM_J}(\alpha_i, \theta_i, \tilde{\gamma}, \tilde{\beta}, \tilde{\alpha}) = \sum_{m'=\epsilon}^{J} y_{KlLm'}^{Jp}(\alpha_i, \theta_i) B_{m'}^{JpM_J}(\tilde{\gamma}, \tilde{\beta}, \tilde{\alpha}). \tag{12}$$

Here, we have introduced the quantum numbers of the grand angular momentum K and those of the usual angular momentum $\mathbf{l} = -i\mathbf{y} \times \nabla_{\mathbf{y}}$, $\mathbf{L} = -i\mathbf{x} \times \nabla_{\mathbf{x}}$; M_J and m' are projections of the total angular momentum $\mathbf{J}$. The body-fixed z-axis is specified by the set $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}$, and $B_{m'}^{JpM_J}$ are the parity preserving combinations of the Wigner D-functions (Matveenko and Fukuda 1996).

In what follows, we shall manipulate with two $\{i = 2\}$ and $\{j = 3\}$ 'physical' (Matveenko and Fukuda 1998) subsets of HH satisfying either $K = J$ (normal parity states) or $K = J + 1$ (abnormal ones) assumption, being equivalent to the conditions introduced earlier by Schwartz (1961)

$$L + l = J \quad \text{if} \quad \{p = (-)^J\} \qquad \text{or} \qquad L + l = J + 1 \quad \text{if} \quad \{p = -(-)^J\}. \tag{13}$$

2. Intrinsic hyperspherical harmonics (IHH)

Using (13) we may note that quantum numbers L and K are not needed to classify the intrinsic part of HH $y_{KlLm'}^{Jp}(\alpha_i, \theta_i)$ (12) and specify the IHH $\|y_l(\alpha_i, \theta_i)\|_{\hat{\omega}}^{Jp}$ as the vector-column solving the matrix differential equation in two variables (Matveenko and Fukuda 1998)

$$\left(\left[\Lambda_i^2\right]_{\hat{\omega}}^{Jp} - K(K+4)\right)\|y_l(\alpha_i, \theta_i)\|_{\hat{\omega}}^{Jp} = 0 \qquad K - J = \epsilon \leqslant l \leqslant J \tag{14}$$

where $\left[\Lambda_i^2\right]_{\hat{\omega}}^{Jp}$ is the hyperangular part of the total three-body kinetic energy operator (1) projected onto the states of fixed total angular momentum J and parity p. With the definition $\epsilon = K - J$ the above eigenvalue equation works for states of any parity. After introducing the auxiliary vector-column $\|p_l(\alpha_i, \theta_i)\|^{Jp}$ for all possible $l = \epsilon, \ldots, J$

$$\|p_l(\alpha_i, \theta_i)\|^{Jp} = \frac{\sin^L\alpha_i \cos^l\alpha_i}{\sqrt{l!!\,L!!}} \begin{pmatrix} (-1)^{\epsilon} U_{\epsilon L}^{Jpl} P_l^{\epsilon}(\theta_i) \\ \cdots \\ (-1)^m U_{mL}^{Jpl} P_l^m(\theta_i) \\ \cdots \\ (-1)^l U_{lL}^{Jpl} P_l^l(\theta_i) \\ 0 \\ \cdots \\ 0 \end{pmatrix} \qquad m = \epsilon, \ldots, l\{J\} \tag{15}$$

we are ready to write the basic entity of the approach, the vector-column IHH $\|y_l(\alpha_i, \theta_i)\|_{\hat{\omega}}^{Jp}$, in the factorized form

$$\|y_l(\alpha_i, \theta_i)\|_{\hat{\omega}}^{Jp} = \mathbf{d}^{Jp}(\omega_i)\|p_l(\alpha_i, \theta_i)\|^{Jp} \tag{16}$$

with an arbitrary choice of the quantization axis within the particle plane (direction $\hat{\boldsymbol{\omega}}$). Here, $\cos\omega_i = \hat{\boldsymbol{\omega}}\cdot\hat{\mathbf{x}}_i$, $\mathbf{d}^{Jp}(\omega_i)$ is the parity preserving combination of the Wigner rotation matrices as was discussed above. In (15) $P_l^m(\theta_i)$ are the normalized associated Legendre polynomials; the coefficients

$$U_{mL}^{Jpl} = p(-)^{J+l+m}\sqrt{2-\delta_{0m}}(l, J, -m, m|L, 0) \qquad \left(\sum_{m=0}^{l}\left(U_{mL}^{Jpl}\right)^2 = 1\right) \tag{17}$$

were defined by Chang and Fano (1972). Though the index m' can be formally bigger than l, the components of (15) for $m' > l$ are equal to zero, which allows one to give the solution matrices composed from $\| p_l(\alpha_i, \theta_i)\|^{Jp}$ $(\hat{\omega}_i = 0)$ in the triangular form.

As we have demonstrated earlier (Matveenko 1999), for each Jacobi channel there exists one more coordinate system, the one with $\hat{\boldsymbol{\omega}} = \hat{\mathbf{y}}_i$, for which case the solution matrix is simple: of the left-upper triangular form (triangular with respect to the second diagonal). It will be $\|\bar{\mathbf{p}}(\alpha_i, \theta_i)\|^{Jp}$, as introduced earlier, and can be calculated either using (10) or the prescription from Matveenko (1999) which reads: starting with the solution matrix $\|\mathbf{p}(\alpha_i, \theta_i)\|^{Jp}$ we first interchange its columns by reflection with respect to the central line, and substitute α_i by $\alpha_i - \pi/2$ and θ_i by $-\theta_i$. Using this rule we actually produced from (6) the example (8) discussed in the introduction.

In our previous paper (Matveenko and Czerwonko 2001) one can find as an example all $(2*2)$ matrices that are involved in the discussion of the normal parity HH having $(J = 1, p = -1)$ symmetry.

3. New hyper-trigonometry identities

Formally, there are only minor details that make the treatment of the normal and abnormal parity cases different though, of course, we get different identities in the two cases. This statement does not work in only one case: the abnormal parity case for the states of $(J = 2, p = -1)$ symmetry just reproduces the results of the normal $(J = 1, p = -1)$ states (we get $(2*2)$ matrices for both cases, the solution matrices are different but some cancellation occurs in (9)) and we arrive at (8).

We shall start with the normal parity case: setting $p = (-)^J$ and using $(p = n)$ index for the parity-dependent expressions. The body-fixed quantization axis $(\hat{\boldsymbol{\omega}} = \hat{\mathbf{y}}_3)$ will be used in order to make the analysis simpler without losing generality. As has already been noted, the corresponding solution matrices originating in $i = \{2\}$ and $i = \{3\}$ Jacobi channels are: $\|\mathbf{y}(\alpha_2, \theta_2)\|_{\hat{\mathbf{y}}}^{Jp} = \mathbf{d}^{Jp}(\theta_{23}'')\|\bar{\mathbf{p}}(\alpha_2, \theta_2)\|^{Jp}$ and $\|\mathbf{y}(\alpha_3, \theta_3)\|_{\hat{\mathbf{y}}}^{Jp} = \|\bar{\mathbf{p}}(\alpha_3, \theta_3)\|^{Jp}$, respectively. Any IHH $\| y_l(\alpha_2, \theta_2)\|_{\hat{\mathbf{y}}}^{Jp}$ can be expressed as a linear combination of $\| y_{l'}(\alpha_3, \theta_3)\|_{\hat{\mathbf{y}}}^{Jp}$, $(\epsilon = 0 \leqslant l, l' \leqslant J)$; the corresponding transformation matrix R^{Jp} has been introduced by Raynal and Revai (1970) for five-dimensional HH in the ordinary space-fixed frame. In our case, for vector-column IHH depending only on two variables (15), the proper relation will be

$$\hat{\mathbf{d}}^{Jn}(\theta_{23}'')\|\bar{\mathbf{p}}(\alpha_2, \theta_2)\|^{Jn} = \|\bar{\mathbf{p}}(\alpha_3, \theta_3)\|^{Jn}\left\| d_{-J/2+l, -J/2+l'}^{J/2}(2\phi_{23})\right\| \qquad (l, l' = 0, \ldots, J) \tag{18}$$

where all involved matrices have the $(J+1)*(J+1)$ dimension.

The left upper triangle matrices $\|\bar{\mathbf{p}}(\alpha_i, \theta_i)\|^{Jn}$ with matrix elements $\bar{p}_{lm}^{Jn}(\alpha_i, \theta_i)$ have their columns numbered by the quantum number of the pair angular momentum l $(0 \leqslant l \leqslant J)$ while its projection m onto the body-fixed $\mathbf{y}_i$ axis serves for row numbering (only $\bar{p}_{lm}^{Jn}(\alpha_i, \theta_i)$ with $0 \leqslant m \leqslant l$ are nonzero). Two more matrices, normal parity rotation matrix $\hat{\mathbf{d}}^{Jn}$ and the usual Wigner rotation matrix $\left\| d_{-J/2+l, -J/2+l'}^{J/2}\right\|$, are orthogonal. We have used in (18) the result derived by Raynal (1972) allowing one to express Raynal–Revai matrices for the lower extreme value of the grand angular momentum $K = J$ in terms of the usual Wigner rotation matrices: $\mathcal{D}^{Jn}(\phi) = \left\| d^{J/2}_{-J/2+l,-J/2+l'}(2\phi) \right\|$, $(0 \leqslant l, l' \leqslant J)$. The coefficients (17), composing (15) for the normal parity case, read

$$U^{Jnl}_{mL} = \left(\frac{2}{1+\delta_{m0}} \frac{(2l)!(2L+1)!(J-m)!(J+m)!}{(2J+1)!(l-m)!(l+m)!(L!)^2} \right)^{1/2} \qquad L = J - l. \tag{19}$$

Using the triangular structure of $\|\bar{\mathbf{p}}(\alpha_i, \theta_i)\|^{Jn}$, and analytic expressions for the matrix elements of the first row (first column) of $\hat{\mathbf{d}}^{Jn}$ (Matveenko 1999) and $\left\| d^{J/2}_{m,m'}(\phi) \right\|$ (Varshalovich *et al* 1998), we present the four simplest scalar identities for the corner matrix elements of the matrix equation (18). For the $[0, 0]$ one (the upper left corner of (18)) we have the most complicated expression; it includes two summations

$$\sum_{m=0}^{J} 2\sqrt{\frac{2}{(1+\delta_{0m})(2J+1)J!!}} P^J_m(\theta''_{23}) \sin^J(\alpha_2) P^J_m(\theta_2) = \sum_{l=0}^{J} \bar{p}^{Jn}_{J-l,0}(\alpha_3, \theta_3) \sqrt{\frac{J!}{l!(J-l)!}} \cos^{J-l}\phi_{23} \sin^l(-\phi_{23}). \tag{20}$$

The $[0, J]$ case is simpler; only one summation is needed

$$\frac{1}{\sqrt{(2J+1)J!!}} \cos^J\alpha_2 P^0_J(\theta''_{23}) = \sum_{l=0}^{J} \bar{p}^{Jn}_{J,J-l}(\alpha_3, \theta_3) \sqrt{\frac{J!}{l!(J-l)!}} \sin^{J-l}\phi_{23} \cos^l\phi_{23}. \tag{21}$$

The case $[J, 0]$ is similar to the $[0, J]$ one, it includes one summation:

$$\frac{(2J-1)!!}{\sqrt{2J!!(2J+1)}} \sin^J\alpha_3 \cos^J\phi_{23} \sin^J(\theta_3) = \sum_{m=0}^{J} d^{Jn}_{Jm}(\theta''_{23}) \bar{p}^{Jn}_{Jm}(\alpha_2, \theta_2). \tag{22}$$

Moreover, the last $[J, J]$ corner provides the simplest result

$$d^{Jn}_{J0}(\theta''_{23}) \bar{p}^{Jn}_{00}(\alpha_2, \theta_2) = \bar{p}^{Jn}_{JJ}(\alpha_3, \theta_3) d^{J/2}_{-J/2-J/2}(2\phi_{23}). \tag{23}$$

Using explicitly (15), see also (Matveenko and Fukuda 1996, Varshalovich *et al* 1998), we get

$$\bar{p}^{Jn}_{00}(\alpha_2, \theta_2) = \cos^J\alpha_2 / (\sqrt{2J!!})$$

$$\bar{p}^{Jn}_{JJ}(\alpha_3, \theta_3) = (2J-1)!! \sqrt{\frac{1}{(2J)!J!!}} \sin^J\alpha_3 \sin^J\theta_3$$

$$d^{Jn}_{J0}(\beta) = \sqrt{\frac{2(2J)!}{(J)!(J)!}} (-1/2\sin\beta)^J$$

$$d^{J/2}_{-J/2-J/2}(2\phi_{23}) = \cos^J(\phi_{23}).$$

After substituting the above expressions into (23) and using

$$a_J = \left(\frac{1}{2}\right)^J \frac{(2J)!}{J!(2J-1)!!} \equiv 1$$

we arrive at the much simpler form

$$(\cos\alpha_2 \sin(-\theta_{23}))^J = (\sin\alpha_3 \sin\theta_3 \sin\phi_{23})^J.$$

It means that (23) is J-independent thus coinciding with the [1, 1] element of (8). Similarly, it can be checked that putting $J = 1$ into equations (20)–(22) one is reproducing the corresponding scalar equalities from the matrix equation (8).

For the abnormal parity states, i.e. IHH defined by the conditions $K = J + 1$ and $p = -(-)^J$, we get similarly the matrix identities interconnecting IHH in the $i = \{2\}$ and $i = \{3\}$ Jacobi channels

$$\hat{\mathbf{d}}^{Ja}(\theta''_{23})\|\bar{\mathbf{p}}(\alpha_2,\theta_2)\|^{Ja} = \|\bar{\mathbf{p}}(\alpha_3,\theta_3)\|^{Ja}\left\|d^{(J-1)/2}_{-(J+1)/2+l,-(J+1)/2+l'}(2\phi_{23})\right\| \qquad (\epsilon = 1 \leqslant l, l' \leqslant J) \tag{24}$$

where now we have to deal with the $(J * J)$ matrices having $1 \leqslant l, m \leqslant J$ numbering columns and rows of $\|\bar{\mathbf{p}}(\alpha_i,\theta_i)\|^{Ja}$ (as in the normal parity case $\bar{p}^{Ja}_{lm}(\alpha_i,\theta_i)$ matrix elements are nonzero only for $1 \leqslant m \leqslant l$). Here, (24), once again we have used the result of Raynal (1973), this time for the lowest possible value of the grand angular momentum $K = J + 1$ in which case $\mathcal{R}^{Ja}(\phi_{23})$ once again can be given as a regular Wigner rotation matrix. The Chang–Fano coefficients (17) are now expressed by

$$U^{Jal}_{mL} = 2Lm\left(2(2L+1)\frac{(2l-1)!(2L-1)!(J-m)!(J+m)!}{(2J+2)!(l-m)!(l+m)!(L!)^2}\right)^{1/2} \qquad L = J - l + 1. \tag{25}$$

The four simplest scalar identities for the corner matrix elements of (23) will read

$$[1,1] \to \sum_{m=1}^{J} d^{Ja}_{1m}(\theta''_{23})\bar{p}^{Ja}_{Jm}(\alpha_2,\theta_2) = \sum_{l=1}^{J}\frac{U^{Ja,J-l+1}_{1,l}\sin^{J-l+1}\alpha_3\cos^l\alpha_3}{\sqrt{(J-l+1)!!(l)!!}}P_{J-l+1}(\theta_3) \times\sqrt{\frac{(J-1)!}{l!(J-l-1)!}}\cos^{J-l-1}\phi_{23}\sin(-\phi_{23})^l \tag{26}$$

$$[1,J] \to d^{Ja}_{11}(\theta''_{23})\frac{\sqrt{3}}{2\sqrt{J!!}}\cos^J\alpha_2\sin\alpha_2\sin\theta_2 = \sum_{l=1}^{J}\bar{p}^{Ja}_{J-l+1,1}(\alpha_3,\theta_3)\sqrt{\frac{(J-1)!}{l!(J-l-1)!}}\cos^l\phi_{23}\sin(-\phi_{23})^{J-l-1} \tag{27}$$

$$[J,1] \to (2J-1)!!\sqrt{\frac{3J}{(J+1)(2J)!J!!}}\cos\alpha_3\sin^J\alpha_3\sin^J\theta_3\cos^{J-1}(\phi_{23}) = \sum_{m=1}^{J} d^{Ja}_{Jm}(\theta''_{23})\bar{p}^{Ja}_{Jm}(\alpha_2,\theta_2) \tag{28}$$

$$[J,J] \to d^{Ja}_{J1}(\theta''_{23})\bar{p}^{Ja}_{11}(\alpha_2,\theta_2) = \bar{p}^{Ja}_{JJ}(\alpha_3,\theta_3)d^{(J-1)/2}_{-(J-1)/2(J-1)/2}(2\phi_{23}). \tag{29}$$

Here again, the $[J, J]$ result is the simplest one and does not depend on J. Actually, using (15) and the prescriptions for building $\bar{p}^{Jp}_{lm}(\alpha_i,\theta_i)$ mentioned above we can get

$$\bar{p}^{Ja}_{11}(\alpha_2,\theta_2) = \sqrt{3}\cos^J\alpha_2\sin\alpha_2\sin\theta_2/(2\sqrt{J!!})$$

$$\bar{p}^{Ja}_{JJ}(\alpha_3,\theta_3) = (2J-1)!!\sqrt{\frac{3J}{(J+1)(2J)!J!!}}\cos\alpha_3\sin^J\alpha_3\sin^J\theta_3$$

and can derive from Matveenko and Fukuda (1996) and Varshalovich *et al* (1998)

$$d^{Ja}_{J1}(\beta) = \sqrt{\frac{(2J)!}{(J+1)!(J-1)!}}(-1/2\sin\beta)^{J-1} \qquad d^{(J-1)/2}_{-(J-1)/2(J-1)/2}(2\phi) = \sin^{J-1}(\phi).$$

On substituting the above results into (29) and using the hyperspherical *sin*-theorem (Matveenko and Czerwonko 2001):

$$\sin\alpha_2\cos\alpha_2\sin\theta_2 = \sin\alpha_3\cos\alpha_3\sin\theta_3$$

we shall just reproduce the normal parity simple case $[J, J]$ (23). Both matrix equalities, (18) and (24), were also checked numerically. For this purpose we have introduced hyperspheroidal coordinates $\xi = (x_1 + x_2)/x_3$, $\eta = (x_1 - x_2)/x_3$ (Matveenko and Fukuda 1996) and expressed in their terms all auxiliary variables entering the above identities. As an example we just note that the channel-independent *hyper-sin* identity then reads

$$\cos\alpha_i\sin\alpha_i\sin\theta_i = \frac{\sqrt{c(\xi^2-1)(1-\eta^2)}}{1 + c(\xi^2+\eta^2-2\kappa\xi\eta+\kappa^2-1)} \qquad (i = 1, 2, 3). \tag{30}$$

Constants c, κ were defined in the introduction.

4. Conclusions

This paper completes our analysis of subsets of hyperspherical harmonics IHH, which, if expressed (for any Jacobi channel) in one of the two available body-fixed quantization axes, are simple vector-column functions in two variables (only associated Legendre polynomials of a special normalization and *sin(cos)*-functions are needed). Any function from the degenerate set of IHH in the $\{i\text{th}\}$-channel can be expressed as a linear combination of IHH in the $\{j\text{th}\}$-channel. It gives an obvious ground for manipulation with classical mathematical functions of different arguments: geometrical angles of the particle triangle, regular hyperspherical angles α_i, θ_1 (4), which depend on both particle masses and interparticle distance, and angles of the kinematic rotation ϕ_{ij} (3) depending on particle masses. For any given value of the total angular momentum of a three-body system the underlying matrix expressions derived in this paper, (18) and (24), provide $(J+1)^2$ and J^2 scalar equalities, respectively. Roughly speaking, all are new, only a few of them can easily be derived using standard trigonometry tools but only the simplest of them may be of some independent analytic value. Matveenko *et al* (2001) have used identities of that kind in realistic three-body calculations, utilizing hyperspherical coordinates.

An example of an analytic use of the derived identities can be found in our recent discussion of the semianalytic description of the highly rotational states of antiprotonic He (Matveenko and Alt 2000). In that paper, we were able to resolve the Coriolis coupling analytically using the angular part of the variational primitives in the form (15). In this case, the matrix structure of the Schrödinger operator (see equation (1)) can be reduced to the calculation of the scalar analytic function ((J, p) pair is fixed)

$$\sigma^{ij}_{ll'}(\xi,\eta) = \sum p_{lm}(\alpha_i,\theta_i)p_{l'm'}(\alpha_j,\theta_j)d^{Jp}_{mm'}(\omega_{ij}) = \sum \bar{p}_{lm}(\alpha_i,\theta_i)\bar{p}_{l'm'}(\alpha_j,\theta_j)d^{Jp}_{mm'}(\bar{\omega}_{ij}) \tag{31}$$

with $\cos\omega_{ij} = \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j$ and $\cos\bar{\omega}_{ij} = \hat{\mathbf{y}}_i \cdot \hat{\mathbf{y}}_j$, respectively. As at the end of section 3, before starting calculations we are to introduce a pair of global hyperspherical angles ξ, η common to all Jacobi channels. Actually, $\sigma^{ij}_{ll'}$ is the matrix element of the 'angular form-factor matrix'

$$\left[\|\mathbf{y}^i\|^{Jp}_{\omega}\right]^T \left[\|\mathbf{y}^j\|^{Jp}_{\omega}\right] \tag{32}$$

which, being a physical quantity, should not depend on the choice of the body fixed quantization axis $\hat{\omega}$. Our new identities just reflect the opportunity of using different coordinate systems for different physical three-body problems.

A peculiar feature of the identities (18) and (24), which has been already mentioned in our previous paper I, is that they relate Wigner rotation matrices of two types: regular ones $\left\| d^J_{mm'}(\theta) \right\|$, as defined by Varshalovich *et al* (1998), and parity-projected ones $\hat{\mathbf{d}}^{Jp}(\theta)$. The latter can be defined in the factorized form, recently derived in a similar context by Matveenko (1999) and by Manakov *et al* (2000) using a different technique.

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